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THE USE OF R AND S^2 CHARTS UNDER NONSTANDARD CONDITIONS

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0. Abstract. R and S^2 charts for monitoring the variability of a process are studied when the control variance, σ_0^2, is unknown, when observations are non-Gaussian, and when the process drifts. Geometric bounds on run-length distributions for the R and S^2 charts are given for all underlying distributions when σ_0^2 is estimated in a base period. Similar bounds apply in the case of a drifting process, the notion of which is developed rigorously. It is shown that normal-theory properties of these charts hold exactly for every spherical process when a suitable estimate for σ_0^2 is used. Run-length distributions otherwise are shown to be ordered stochastically given a peakedness ordering on the underlying spherical processes.

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Key words and phrases. Monitoring process variance, run-length distributions, non-Gaussian errors, spherical processes, stochastic ordering.

<u>1. Introduction</u>. Standard procedures for monitoring the variability of a production process are the R and S^2 charts. Let $\{\underset{\sim}{X}_i = (X_{i1}, X_{i2}, \ldots, X_{in}); i = 1, 2, \ldots\}$ be the outcomes and $\{[X_{i(1)} \leq X_{i(2)} \leq \cdots \leq X_{i(n)}]; i = 1, 2, \ldots\}$ the ordered outcomes in a succession of samples of n observations each; let σ^2 be the process variance and σ_0^2 its value when the process is in control; and let $R_i = (X_{i(n)} - X_{i(1)})$ and $S_i^2 = \sum_{j=1}^n (X_{ij} - \bar{X}_i)^2/(n-1)$ respectively be the typical sample range and sample variance. The R and S^2 charts are graphs of the standardized ratios $\{R_i/\sigma_0; i = 1, 2, \ldots\}$ and $\{S_i^2/\sigma_0^2; i = 1, 2, \ldots\}$ plotted against time on the horizontal scale. The monitored process is taken to be in control as long as the ratios are within their control limits, i.e., $R_i/\sigma_0 \leq c_\alpha$ or $S_i^2/\sigma_0^2 \leq d_\alpha$, as appropriate. Otherwise the chart signals that the process is not in control and that corrective action is needed. The object is to detect early a shift in the variance, yet to signal infrequently when the process is in control. The sample number N at which a signal occurs is called the run length; its distribution determines the operating characteristics of the R and S^2 charts. The mean of the run-length distribution, if defined, is called the <u>average run length</u>.

Usually it is assumed that σ_0^2 is known; that independent random samples are generated on successive occasions by a Gaussian process; and, for evaluating run-length distributions, that σ^2 remains constant throughout the monitoring period. Then the run-length distributions for the R and S^2 charts are geometric, their parameters being the probability of exceeding the control limit on any particular occasion. Although S^2 is more efficient than R, the latter claims wide usage in statistical quality control owing to its greater simplicity and the fact that loss in efficiency may be compensated by larger samples in nondestructive testing.

In routine use one or more of the usual assumptions may fail. When σ_0^2

is unknown a common practice is to estimate it during a base-line period when the process is in control, and to maintain the R and S^2 charts using this estimate in place of σ_0^2. The run-length distributions, no longer geometric, are complicated by stochastic dependence among the resulting ratios. Properties of the actual run-length distributions are examined in Section 2 using a variety of estimates for σ_0^2 without stipulating the underlying distribution. Even when σ_0^2 is known the run-length distributions are not geometric unless σ^2 remains constant. The notion of a drifting process is developed precisely in Section 2 and stochastic bounds for its run-length distributions are given. When Gaussian assumptions fail little is known beyond the usual caution that normal-theory procedures for variances are not robust. In Section 3 the robustness of normal-theory R and S^2 charts is studied for a large class of underlying processes. Section 4 gives some numerical comparisons of practical interest stemming from earlier sections, with special reference to normal-theory R and S^2 charts.

2. Bounds on Run-length Distributions. Given a sample $\underset{\sim}{X}_0 = (X_{01}, \ldots X_{0m})$ of m observations taken while the process is in control, let V_0 be an estimate for σ_0. In particular, V_0 may be the sample range R_0, the sample standard deviation S_0, the inter-quartile range, or some other estimate which need not be stipulated here. Denote by N_R and N_S the run lengths of the R and S^2 charts, and by $F_R(t)$ and $F_S(t)$ their respective cumulative distribution functions (cdf's). When the process variance remains constant throughout the monitoring period these distributions are determined by the Studentized ratios $\{R_i/V_0;\ i = 1, 2, \ldots\}$ and $\{S_i^2/V_0^2;\ i = 1, 2, \ldots\}$ as follows, where $P(E|v)$ and $P(E)$ are conditional and unconditional probabilities of the event E. It is clear for each positive integer t that

$$P(N_R > t) = P(R_1/V_0 \leq c_\alpha, \ldots, R_t/V_0 \leq c_\alpha) \\ = \int_0^\infty [P(R \leq vc_\alpha|v)]^t dG(v) \tag{2.1}$$

$$P(N_S > t) = P(S_1^2/V_0^2 \leq d_\alpha, \ldots, S_t^2/V_0^2 \leq d_\alpha) \\ = \int_0^\infty [P(S^2 \leq v^2 d_\alpha|v)]^t dG(v) \tag{2.2}$$

where R and S^2 have the same distributions as R_i and S_i^2, and $G(\cdot)$ is the cdf of V_0. Moreover, in support of later developments using finite-dimensional distribution theory, we note that $P(N_R < \infty) = 1 = P(N_S < \infty)$, i.e., both charts eventually signal with unit probability. For R charts this follows on writing

$$P(N_R = t) = \int_0^\infty [P(R \leq vc_\alpha|v)]^{t-1}[P(R > vc_\alpha|v)]\, dG(v); \tag{2.3}$$

on noting that $H(t;v) = [P(R \leq vc_\alpha|v)]^{t-1}[P(R > vc_\alpha|v)] < 1$, that $\int_0^\infty dG(v) = 1$, and that $H(t;v) \to 0$ as $t \to \infty$ for each $v \geq 0$; and on applying the dominated convergence theorem to infer that

$$\lim_{t \to \infty} P(N_R = t) = \int_0^\infty \lim_{t \to \infty} H(t;v)dG(v) = 0. \tag{2.4}$$

Similar arguments establish that $P(N_S < \infty) = 1$.

Under Gaussian assumptions a survey of the relevant distribution theory follows. With $V_0 = S_0$ the probability $P(N_R > t)$ is determined by the joint distribution of Studentized ranges whose marginals are of the type considered and tabulated by Pachares (1959), Harter (1960), and Owen (1962), although the joint distribution appears not to have been studied. Similarly $P(N_S > t)$ derives from the known Studentized largest chi-squared distribution considered by Armitage and Krishnaiah (1964), David (1956), Finney (1941), Ghosh (1955),

Gupta (1963), Nair (1948), and Ramachandran (1956). Under Gaussian assumptions with $V_0 = R_0$, $P(N_R > t)$ is determined by the joint distribution of $\{R_1/R_0, \ldots, R_t/R_0\}$, apparently not considered before, each marginal of which is of the type studied by Link (1950). To evaluate (2.1) and (2.2) in particular cases, recourse may be taken to numerical methods; unfortunately, even the extensive tables of Armitage and Krishnaiah (1964) are tabulations of upper percentiles limited to the range $1 \leq t \leq 12$ and thus are not suited to evaluating $P(N_S > t)$.

Rather than treat normal-theory distributions further, we turn instead to general structural properties of $F_R(t)$ and $F_S(t)$ which depend neither on Gaussian assumptions nor the particular choice for V_0. To this end let $G(t;\alpha)$ be the cdf of the geometric distribution having the parameter α. Our first result provides bounds on the actual run-length distributions $F_R(t)$ and $F_S(t)$ in terms of $G(t;\alpha)$.

<u>THEOREM 2.1.</u> Let $\{\underset{\sim}{X}_0, \underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_t\}$ be mutually independent sample outcomes such that $\{\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_t\}$ are distributed identically. Choose c_α and d_α such that $P(R_i/V_0 \leq c_\alpha) = 1 - \alpha = P(S_i^2/V_0^2 \leq d_\alpha)$, and let N have the distribution $G(t;\alpha)$. Then N_R and N_S are stochastically larger then N whatever be the underlying distribution and choice for V_0, i.e., $G(t;\alpha) \geq F_R(t)$ and $G(t;\alpha) \geq F_S(t)$ for every $t > 0$.

<u>Proof.</u> Consider sequences $\{R_i/V_{0i};\ i = 1, 2, \ldots\}$ and $\{S_i^2/V_{0i}^2;\ i = 1, 2, \ldots\}$ of independent ratios having the same distributions as R_i/V_0 and S_i^2/V_0^2, respectively, and let N_R^* and N_S^* be run lengths associated with these. Because N_R^* and N_S^* clearly are geometric, it suffices to show that $P(N_R > t) \geq P(N_R^* > t)$ and $P(N_S > t) \geq P(N_S^* > t)$ for each fixed t. Consider some

function $\phi(\underset{\sim}{X}_i, \underset{\sim}{X}_0) \geq 0$ and suppose $\underset{\sim}{X}$ has the distribution of $\underset{\sim}{X}_i$. Under the hypothesis of the theorem it is known (cf. Khatri (1967), for example) that

$$P(\phi(\underset{\sim}{X}_1,\underset{\sim}{X}_0) \leq c, \ldots, \phi(\underset{\sim}{X}_t,\underset{\sim}{X}_0) \leq c) \geq [P(\phi(\underset{\sim}{X},\underset{\sim}{X}_0) \leq c]^t. \tag{2.5}$$

On choosing ϕ successively as $\phi(\underset{\sim}{X}_i,\underset{\sim}{X}_0) = R_i/V_0$ and $\phi(\underset{\sim}{X}_i,\underset{\sim}{X}_0) = S_i^2/V_0^2$ and applying (2.5) to (2.1) and (2.2), we find that $P(N_R > t) \geq [P(R/V_0 \leq c_\alpha)]^t = P(N_R^* > t)$ and $P(N_S > t) \geq P(N_S^* > t)$, as required.

Several consequences of Theorem 2.1 are immediate. One is that average run lengths for the R and S^2 charts, when σ_0^2 is estimated, are greater than those of the approximating geometric distributions. Of considerable practical interest is the error incurred on approximating $F_R(t)$ and $F_S(t)$ by $G(t;\alpha)$; this topic is studied numerically in Section 4 for the Gaussian case.

To make further progress we suppose the underlying distributions belong to a scale-parameter family. Let $\{\sigma_0^2, \sigma_1^2, \ldots\}$ be a sequence of scale parameters corresponding to successive samples and, with $\gamma_i = \sigma_i^2/\sigma_0^2$, let $\underset{\sim}{\gamma} = (\gamma_1,\gamma_2, \ldots)$. Recall that expressions (2.1), (2.2) and Theorem 2.1 apply when $\{\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_t\}$ are distributed identically; under a scale-parameter family this corresponds either to the case $\sigma_i^2 = \sigma_0^2$, when the process is in control, or to the case $\sigma_i^2 = \sigma^2$ for all i and some $\sigma^2 > \sigma_0^2$. To treat the general case, here called a <u>drifting process</u>, let Γ consist of all sequences $\underset{\sim}{\gamma} = (\gamma_1, \gamma_2, \ldots)$ for which $\gamma_i \geq 1$ for all i. Two such sequences are said to be ordered as $\underset{\sim}{\gamma} \prec \underset{\sim}{\gamma}'$ whenever $\gamma_i \leq \gamma_i'$ for all i. Under this incomplete ordering an element $\underset{\sim}{\gamma}_m \in \Gamma_0$ is said to be <u>minimal</u> for $\Gamma_0 \subset \Gamma$, and $\underset{\sim}{\gamma}_M \in \Gamma_0$ to be <u>maximal</u> for Γ_0, if, for every $\underset{\sim}{\gamma} \in \Gamma_0$, $\underset{\sim}{\gamma}_m \prec \underset{\sim}{\gamma} \prec \underset{\sim}{\gamma}_M$. In particular, $\underset{\sim}{\gamma}_m = (1, 1, \ldots)$ is minimal for Γ.

As the run-length distributions for drifting processes generally depend

on γ, we write the typical cdf $F(t)$ as $F(t;\gamma)$ and consider the family $\{F(t;\gamma);\ \gamma \in \Gamma\}$. Such a family is said to be stochastically decreasing in γ if, for γ and γ' in Γ, the ordering $\gamma \prec \gamma'$ implies the stochastic ordering $F(t;\gamma') \geq F(t;\gamma)$ for every $t > 0$. That the families $\{F_R(t;\gamma);\ \gamma \in \Gamma\}$ and $\{F_S(t;\gamma);\ \gamma \in \Gamma\}$ of run-length distributions are stochastically decreasing in γ, and thus that the R and S^2 charts tend to signal more frequently as the process drifts further from control, is established in the following theorem for scale-parameter families of distributions.

THEOREM 2.2. Let $\{F_R(t;\gamma);\gamma \in \Gamma\}$ and $\{F_S(t;\gamma);\gamma \in \Gamma\}$ be families of run-length distributions for the R and S^2 charts when $\{X_0, X_1, \ldots\}$ are independent samples from a scale-parameter family. Then these run-length distributions are stochastically decreasing in γ.

Proof. Consider $F_R(t;\gamma)$, the arguments for $F_S(t;\gamma)$ being identical. Corresponding to (2.1) let $F_\gamma(u_1, \ldots, u_t)$ be the joint cdf of $\{R_1/V_0, \ldots, R_t/V_0\}$ in the general case; let $F_1(u_1, \ldots, u_t)$ be the case that $\gamma_i \equiv 1$ for all i; and write $P_\gamma(N_R > t) = F_\gamma(c_\alpha, \ldots, c_\alpha)$. By independence and the assumption of a scale-parameter family, we infer that

$$P_\gamma(N_R > t) = F_1(\gamma_1^{-1}c_\alpha, \ldots, \gamma_t^{-1}c_\alpha). \tag{2.6}$$

The proof is completed on noting that $P_\gamma(N_R > t)$ is a decreasing function of $(\gamma_1, \gamma_2, \ldots)$ and thus $F_R(t;\gamma)$ is an increasing function, as asserted.

Run-length distributions, depending on an excess of parameters, are complicated in the case of drifting processes even when σ_0^2 is known. Under further conditions on γ Theorem 2.2 supports the construction of envelopes of curves containing $F_R(t;\gamma)$ and $F_S(t;\gamma)$ in terms of distributions of types (2.1) and (2.2). Suppose $\gamma = (\gamma_1, \gamma_2, \ldots)$ is a bounded sequence having

$\gamma_m = \inf(\gamma_1, \gamma_2, \ldots)$ and $\gamma_M = \sup(\gamma_1, \gamma_2, \ldots)$, and let $\underset{\sim}{\gamma}(m) = \gamma_m(1, 1, \ldots)$ and $\underset{\sim}{\gamma}(M) = \gamma_M(1, 1, \ldots)$. Clearly $F_R(t;\underset{\sim}{\gamma}(m))$ and $F_R(t;\underset{\sim}{\gamma}(M))$ are distributions of type (2.1), and $F_S(t;\underset{\sim}{\gamma}(m))$ and $F_S(t;\underset{\sim}{\gamma}(M))$ are of type (2.2), these being geometric when σ_0^2 is known. Bounds on $F_R(t;\underset{\sim}{\gamma})$ and $F_S(t;\underset{\sim}{\gamma})$ for drifting processes are given explicitly in the following theorem for any scale-parameter family of underlying distributions.

<u>THEOREM 2.3.</u> Let $F_R(t;\underset{\sim}{\gamma})$ and $F_S(t;\underset{\sim}{\gamma})$ be run-length distributions for the R and S^2 charts of a drifting process having the bounded parameter sequence $\underset{\sim}{\gamma} = (\gamma_1, \gamma_2, \ldots)$. If $\{\underset{\sim}{X}_0, \underset{\sim}{X}_1, \ldots\}$ are independent samples from a scale-parameter family, then both $F_R(t;\underset{\sim}{\gamma})$ and $F_S(t;\underset{\sim}{\gamma})$ satisfy inequalities of the type

(i) $F(t;\underset{\sim}{\gamma}(M)) \geq F(t;\underset{\sim}{\gamma}) \geq F(t;\underset{\sim}{\gamma}(m))$

for every $t > 0$. In particular, when σ_0^2 is known it is true that

(ii) $G(t;\alpha(M)) \geq F(t;\underset{\sim}{\gamma}) \geq G(t;\alpha(m))$

where $\alpha(m)$ and $\alpha(M)$ respectively are the probabilities of exceeding the control limits on any particular occasion when $\gamma_i = \gamma_m$ and when $\gamma_i = \gamma_M$ for all i.

<u>Proof.</u> The theorem follows directly from the monotonicity properties of $F_R(t;\underset{\sim}{\gamma})$ and $F_S(t;\underset{\sim}{\gamma})$ established in Theorem 2.2, together with the ordering $\underset{\sim}{\gamma}(m) \prec \underset{\sim}{\gamma} \prec \underset{\sim}{\gamma}(M)$ under the condition that $\underset{\sim}{\gamma}$ be bounded.

Several conclusions follow immediately. Because $\underset{\sim}{\gamma}_0 = (1, 1, \ldots)$ is minimal in Γ, it follows for scale-parameter families that $P_\gamma(N_R < \infty) = 1 = P_\gamma(N_S < \infty)$ and thus the R and S^2 charts eventually signal with unit probability whatever be the parameter sequence in Γ. Moreover, the lower and upper bounds of Theorem 2.3 become tighter as the process drifts between narrowing limits,

i.e., as $\gamma_M - \gamma_m$ decreases. The extent of this tightening depends on the underlying distributions and is studied numerically in Section 4 with reference to normal-theory R and S^2 charts. We finally note that, if Γ_0 consists of all sequences γ for which $\gamma(m)$ is minimal and $\gamma(M)$ is maximal, then the given bounds apply uniformly to all members of the family $\{F(t;\gamma);\ \gamma \in \Gamma_0\}$.

3. Robust Aspects of R and S^2 Charts. In practice R and S^2 charts typically are based on Gaussian assumptions together with the independence of $\{X_0, X_1, \ldots\}$. Often neither assumption is tenable in sampling from a continuing production process. Here the weaker assumption of spherical symmetry is made, allowing at once dependencies among observations and a large class of alternative distributions. We suppose first that successive observations are generated by a spherical process and thus are dependent in all except the Gaussian case. An alternative model, for use when successive samples are known to be independent, replaces the normality of each sample of size n by n-dimensional spherical symmetry.

A distribution $L(X)$ on the N-dimensional Euclidean space R^N is said to be spherical if, for each (N x N) orthogonal matrix Q, $L(QX) = L(X)$. Each such X admits the representation $X = RZ$ in polar form such that $L(X)$ is determined completely by the radial distribution $L(R)$ on $(0,\infty)$, and Z is distributed uniformly on the surface of the unit sphere in R^N independently of R. Let $S_N(\theta, I_N)$ be the class of N-dimensional spherical laws symmetric about $\theta \in R^N$, i.e., if $L(Y) \in S_N(\theta, I_N)$, then $L(Y - \theta)$ is spherical; let M be a linear subspace of R^N; and call a function $\phi: R^N \to R^k$ as location-invariant with respect to M if, for every $\xi \in M$ and $X \in R^N$, $\phi(X + \xi) = \phi(X)$. The following result is basic to our subsequent developments.

LEMMA 3.1. Let $L(\underset{\sim}{Y})$ be spherically symmetric about $\underset{\sim}{\theta} \in M$, and let ϕ: $R^N \to R^k$ be any location-invariant function homogeneous of degree zero. Then the distribution $L(\phi(\underset{\sim}{Y}))$ on R^k is invariant for all distributions $L(\underset{\sim}{Y})$ in the class $\{S_N(\underset{\sim}{\theta}, \underset{\sim}{I}_N);\ \theta \in M\}$.

Proof. By the location-invariance of $\phi(\cdot)$ and the fact that M is a subspace of R^N, we may assume that $\underset{\sim}{\theta} = \underset{\sim}{0}$. Then $\underset{\sim}{Y}$ admits the polar representation $\underset{\sim}{Y} = R\underset{\sim}{Z}$. We have $L(\phi(\underset{\sim}{Y})) = L(\phi(R\underset{\sim}{Z}))$ and, by homogeneity, $L(\phi(R\underset{\sim}{Z})) = L(\phi(\underset{\sim}{Z}))$. The fact that $\underset{\sim}{Z}$ is uniform on the surface of the unit sphere, independently of the particular $L(\underset{\sim}{Y}) \in S_N(\underset{\sim}{\theta}, \underset{\sim}{I}_N)$, assures invariance of the distribution $L(\phi(\underset{\sim}{Y}))$ as asserted.

Let N be countable and let $\{U(t);\ t \in N\}$ be a stochastic process such that, for each N and $\{t_1, \ldots, t_N\} \in N$, the joint distribution of $[U(t_1), \ldots, U(t_N)]$ is spherical; then $\{U(t);\ t \in N\}$ is called a spherical process. It is known (cf. Hartman and Wintner (1940), for example) that a countable process is spherical if and only if it is a scale mixture of spherical Gaussian processes, i.e., if for each N and $\{t_1, \ldots, t_N\} \in N$, the joint probability density function (pdf) of $[U(t_1), \ldots, U(t_N)]$ is given by

$$f(u_1, \ldots, u_N) = (2\pi)^{-N/2} \int_0^\infty \tau^{-N} e^{-\underset{\sim}{u}'\underset{\sim}{u}/2\tau^2} dG(\tau) \tag{3.1}$$

where $u = (u_1, \ldots, u_N)$ and $G(\cdot)$ is some cdf on $(0,\infty)$. Various choices for $G(\cdot)$ yield as special cases the spherical Gaussian process, a spherical Cauchy process whose marginals are spherical Cauchy laws, and other spherical stable processes as further examples.

Against this background we consider the performance of normal-theory R and S^2 charts when, in fact, the distribution of $\{\underset{\sim}{X}_0, \underset{\sim}{X}_1, \ldots\}$ exhibits spherical symmetry. Partition N into the disjoint subsets $\{N_0, N_1, \ldots\}$

corresponding to successive sampling periods; let $\mu(t)$ and $\sigma(t)$ be functions on N such that $\sigma(t) > 0$; and model the process $\{X(t);\ t \in N\}$ as $X(t) = \sigma(t)U(t) + \mu(t)$ such that $\{U(t);\ t \in N\}$ is a spherical process. We assume that $\mu(t)$ and $\sigma(t)$ are step functions, constant within samples with jumps occurring between samples, as it might happen when periods of intense sampling activity are followed by long interim periods and the functions are slowly varying. Accordingly let $\mu_i = \mu(t)$ and $\sigma_i = \sigma(t)$ for $t \in N_i$, $i = 0, 1, 2, \ldots$, and let $\underset{\sim}{\gamma} = (\gamma_1, \gamma_2, \ldots)$ such that $\gamma_i = \sigma_i^2/\sigma_0^2 \geq 1$. Suppose for the present that σ_0^2 is not known and that some estimate V_0^2 from a base-line period is used, and denote by $G_R(t;\underset{\sim}{\gamma})$ and $G_S(t;\underset{\sim}{\gamma})$ the actual run-length distributions of the R and S^2 charts under Gaussian assumptions. Under these conditions normal-theory procedures are more than robust; they remain exact for all spherical processes as shown in the theorem following.

<u>THEOREM 3.1.</u> Suppose $\{\underset{\sim}{X}_0, \underset{\sim}{X}_1, \ldots\}$ are generated from a spherical process $\{U(t);\ t \in N\}$ as $\{X_i(t) = \sigma_i U(t) + \mu_i;\ t \in N_i\}$ for $i = 0, 1, 2, \ldots$. If σ_0 is estimated using a location-invariant function V_0 of $\underset{\sim}{X}_0$, then the run-length distributions of the R and S^2 charts are identical to their normal-theory forms $G_R(t;\underset{\sim}{\gamma})$ and $G_S(t;\underset{\sim}{\gamma})$, respectively, whatever be the underlying spherical process $\{U(t);\ t \in N\}$.

<u>Proof.</u> For fixed t let $N = m + tn$ be the number of observations taken up to the point that a signal occurs, and let $\underset{\sim}{Y}' = (\underset{\sim}{Y}_0', \underset{\sim}{Y}_1', \ldots, \underset{\sim}{Y}_t')$ such that $\underset{\sim}{Y}_i = \sigma_i^{-1}\underset{\sim}{X}_i$. Then $L(\underset{\sim}{Y}) \in S_N(\underset{\sim}{\theta},\underset{\sim}{I}_N)$ for $\underset{\sim}{\theta}$ belonging to a subspace M of R^N, and observe that $\phi(\underset{\sim}{Y}) = (\gamma_1^{-1}R_1/V_0, \ldots, \gamma_t^{-1}R_t/V_0)$ not only is location-invariant with respect to M, but is homogeneous of degree zero. From Lemma 3.1 it follows that the joint distribution of $(\gamma_1^{-1}R_1/V_0, \ldots, \gamma_t^{-1}R_t/V_0)$, and thus of $(R_1/V_0, \ldots, R_t/V_0)$, is invariant for all $L(\underset{\sim}{Y}) \in S_N(\underset{\sim}{\theta},\underset{\sim}{I}_N)$. The joint

distribution of $(R_1/V_0, \ldots, R_t/V_0)$ thus is independent of the particular spherical process generating the observations and, as the spherical Gaussian process belongs to the class, the run-length distribution $P_{\gamma}(N_R \le t)$ is identical to its normal-theory form, namely, $G_R(t;\underset{\sim}{\gamma})$. Similar arguments establish invariance properties of S^2 charts.

It may be noted that normal-theory properties of the run-length distributions $G_R(t;\underset{\sim}{\gamma})$ and $G_S(t;\underset{\sim}{\gamma})$ were established in the preceding section under the assumption that $\{\underset{\sim}{X}_0, \underset{\sim}{X}_1, \ldots\}$ are mutually independent. Theorem 3.1 assures that these properties continue to hold for all spherical processes despite dependencies among successive samples, as long as V_0 from a base-line period is used to estimate σ_0. In contrast to these results, the run-length distributions of the R and S^2 charts do depend on the underlying spherical process when σ_0 is known and the base-line period thus remains unobserved. Despite dependencies among successive samples, stochastic bounds for run-length distributions analogous to those of Theorem 2.1 are given as follows for spherical processes.

<u>THEOREM 3.2.</u> Suppose $\{\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots\}$ are generated from a spherical process for which σ_0^2 is known and σ^2 remains constant, and let N_R and N_S be run lengths for the R and S^2 charts having the respective cdf's $F_R(t)$ and $F_S(t)$. Choose $\alpha_R = P(R/\sigma_0 > c_\alpha)$ and $\alpha_S = P(S^2/\sigma_0^2 > d_\alpha)$, where R and S^2 have the same distributions as R_i and S_i^2 for $i = 1, 2, \ldots$. Let N_R^* and N_S^* have the geometric distributions $G(t; \alpha_R)$ and $G(t; \alpha_S)$. Then N_R and N_S respectively are stochastically larger than N_R^* and N_S^*, i.e.,

$$G(t;\alpha_R) \ge F_R(t), \qquad G(t; \alpha_S) \ge F_S(t)$$

for all $t > 0$.

<u>Proof.</u> We supply details for R charts; the proof for S^2 charts is identical using appropriate notation. On representing a spherical process as a scale mixture of Gaussian processes, we infer that the actual run-length distribution can be expressed as the scale mixture

$$P(N_R > t) = \int_0^\infty P(R_1/\tau\sigma_0 \leq c_\alpha, \ldots, R_t/\tau\sigma_0 \leq c_\alpha | \tau) dG(\tau) \tag{3.2}$$

where $G(\cdot)$ is the mixing distribution. Because $\{R_1, \ldots, R_t\}$ are conditionally independent and identically distributed, we have

$$\begin{aligned} P(N_R > t) &= \int_0^\infty [P(R/\tau\sigma_0 \leq c_\alpha | \tau)]^t dG(\tau) \\ &\geq [P(R/\sigma_0 \leq c_\alpha)]^t = P(N_R^* > t) \end{aligned} \tag{3.3}$$

as in the proof of Theorem 2.1, where N_R^* has the distribution $G(t;\alpha_R)$.

Not only do the run-length distributions of R and S^2 charts depend on the underlying spherical process, but the geometric bounds of Theorem 3.2 are similarly dependent through α_R and α_S. For these bounds to be useful it thus would appear that the form of the spherical process must be known in each particular case. Our next developments show that run-length distributions are ordered stochastically whenever the underlying distributions are ordered in their degrees of peakedness. It follows that geometric bounds for a reference distribution, such as the Gaussian law, then apply uniformly for all spherical laws more peaked than the reference distribution.

Let $\mu(\cdot)$ and $\nu(\cdot)$ be measures on R^N; $\mu(\cdot)$ is said to be <u>more peaked</u> about $\underset{\sim}{\theta} \in R^N$ than $\nu(\cdot)$ if, for every compact convex set $E \subset R^N$ symmetric about $\underset{\sim}{\theta}$ under reflection, $\mu(E) \geq \nu(E)$ (cf. Sherman (1955)). If $\mu(\cdot)$ and $\nu(\cdot)$ are finite-dimensional measures characterizing two zero-mean spherical processes, then the μ-process is said to be more peaked about zero than the ν-process if $\mu(\cdot)$ is more peaked about $\underset{\sim}{0} \in R^N$ than $\nu(\cdot)$ for every $N < \infty$. Our basic result is the following.

<u>THEOREM 3.3.</u> Let $F_\omega(t;\underset{\sim}{\gamma})$, with paremeter sequence $\underset{\sim}{\gamma} = (\gamma_1, \gamma_2, \ldots)$, be the

run-length distribution for either the R or S^2 chart of a drifting process generated from a spherical process having the measure $\omega(\cdot)$. Of two processes having the measures $\mu(\cdot)$ and $\nu(\cdot)$, suppose $\mu(\cdot)$ is more peaked than $\nu(\cdot)$. Then the run-length distributions for the R and S^2 charts from the two processes are ordered stochastically as

$$F_\nu(t;\underset{\sim}{\gamma}) \geq F_\mu(t;\underset{\sim}{\gamma})$$

for each fixed $\underset{\sim}{\gamma}$ and $t > 0$.

Proof. Let N_R be the run length of the R chart; S^2 charts are treated similarly. Let $N = tn$ and, as in the developments preceding Theorem 3.1, consider $\{\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_t\}$ and $\{\underset{\sim}{Y}_1, \underset{\sim}{Y}_2, \ldots, \underset{\sim}{Y}_t\}$ such that $\underset{\sim}{Y}_i = \gamma_i^{-1}\underset{\sim}{X}_i$, and let $R(\underset{\sim}{Y}_i)$ be the range function $R(\underset{\sim}{Y}_i) = Y_{i(n)} - Y_{i(1)}$. As the sample range and sample variance are invariant under translation, we may take the process to be symmetric about zero. Let

$$A_i = \{\underset{\sim}{Y} \in R^N \mid R(\underset{\sim}{Y}_i) \leq \gamma_i^{-1}c_\alpha\}; \quad i = 1, 2, \ldots, t$$

and $A = \bigcap_{i=1}^{t} A_i$. Under the typical measure $\omega(\cdot)$ on R^N and the induced measure $P_\omega(\cdot)$ on R^1 we infer that

$$P_\omega(N > t) = \omega(A).$$

Because $R(\underset{\sim}{Y}_i)$, the sum of convex functions, is convex and symmetric, it follows that A_i is a cylinder set in R^N symmetric about $\underset{\sim}{0}$ and having compact convex sections. Therefore A is a compact convex symmetric set and, under the hypothesis that $\mu(\cdot)$ is more peaked about $\underset{\sim}{0} \in R^N$ than $\nu(\cdot)$, it follows that $\mu(A) \geq \nu(A)$, i.e.,

$$P_\mu(N_R > t) \geq P_\nu(N_R > t)$$

which is equivalent to the assertion of the theorem.

4. Numerical Comparisons.

We study numerically some bounds developed in earlier sections, first considering stochastic bounds on $F_R(t)$ and $F_S(t)$,

as given in Theorem 2.1, when the underlying distributions are Gaussian. We treat the case that $n = 10$ observations are taken on each occasion, σ_0^2 having been estimated using the sample variance, S_{0m}^2, of a variable number, m, of observations in a base-line period. To monitor at the level $\alpha = 0.05$, we choose the critical values $c_\alpha(m)$ and $d_\alpha(m)$ from standard tables such that $P(R_i/S_{0m} \le c_\alpha(m)) = 0.95 = P(S_i^2/S_{0m}^2 \le d_\alpha(m))$. We assume that the process is under control, i.e., $\sigma_i^2 = \sigma_0^2$ for all i. The run-length distributions for the R and S^2 charts are found by numerical integration from normal-theory versions of expressions (2.1) and (2.2) using an IMSL (1979) subroutine. Results of these computations are tabulated in Table 1 for R charts and in Table 2 for S^2 charts for various choices of t and m. We report results up to $t = 100$ and $m = 40$, as computational errors are greater for large values of t and m owing to the configuration of the system used. The entry $m = \infty$ in Tables 1 and 2 corresponds to the case that σ_0^2 is known, the base period remains unobserved, and each run-length distribution is geometric having the parameter $\alpha = 0.05$. The distribution $G(t;\alpha)$ also is the limit of $F_R(t)$ and $F_S(t)$ as $m \to \infty$, whatever be the choice for α.

These tabulations suggest a further ordering, namely, that N_R and N_S decrease stochastically as m increases, reflecting successively weaker dependencies among the ratios $\{R_1/S_{0m}, \ldots, R_t/S_{0m}\}$ and $\{S_1^2/S_{0m}^2, \ldots, S_t^2/S_{0m}^2\}$. The geometric approximation is seen to be quite close even for $m = 40$. In addition, Tables 1 and 2 suggest that the R chart may signal more frequently than the S^2 chart when the process is in control, its variance is estimated using S_{0m}^2 from a base period, and both charts operate at the level $\alpha = 0.05$.

We next study examples of the drifting processes considered in Section 2, specifically, the Gaussian case in which successive samples are independent.

We suppose that $n = 10$ observations are drawn on each occasion, including the base period which yields the estimate S_0^2 for σ_0^2. Again c_α and d_α are taken from standard tables with $\alpha = 0.05$. To study the bounds of Theorem 2.3 numerically, we consider the run-length distributions $F_R(t;\underset{\sim}{\gamma})$ and $F_S(t;\underset{\sim}{\gamma})$ associated with parameter sequences of the special type $\underset{\sim}{\gamma}(\xi) = (\xi^2, \xi^2, \ldots)$, where $\xi = \sigma_i/\sigma_0$. Tables 3 and 4 give probabilities from these distributions for ξ taking values in the set $\{1.0, 1.2, 1.5, 2.0\}$, entries having been computed by numerical integration using an IMSL (1979) subroutine. These values in part are presented graphically in Figures 1 and 2, which provide envelopes for the run-length distributions of certain families of drifting processes. For example, all drifting processes which satisfy $\sigma_i/\sigma_0 \leq 1.2$ for all i will have run-length distributions lying in the region bounded by the curves labeled $\xi = 1.0$ and $\xi = 1.2$. Note that, while the run-length distributions for drifting processes depend on the path of the drift, namely, $\underset{\sim}{\gamma} = (\gamma_1, \gamma_2, \ldots)$, the envelopes depend only on the maximum and minimum drifts of the process.

Turning to the spherical processes discussed in Section 3, we note by Theorem 3.2 that geometric bounds are available for the run-length distributions of the R and S^2 charts, the parameters α_R and α_S depending on the particular process. It is instructive to study disturbances in α_R and α_S owing to misspecification of the underlying process. Here we compare the actual values with their nominal values, the control limits c_α and d_α having been set under the mistaken assumption that the process is Gaussian, when, in fact, successive observations are generated from a spherical Cauchy process. These comparisons are given in Table 5 for R charts and in Table 6 for S^2 charts using samples of sizes $n = 3$, 5, and 10. Table 5 was generated by

simulating $P(R/\sigma_0 \leq c_\alpha)$, c_α being the normal-theory control limit, using 100,000 replications for $n = 3$ and $n = 5$ and using 10,000 replications for $n = 10$. Table 6 was constructed on computing $P(S^2/\sigma_0^2 \leq d_\alpha)$, d_α being the normal-theory control limit, using the fact that S^2/σ_0^2 has an F-distribution with $n - 1$ and 1 degrees of freedom in the case of a spherical Cauchy process. The evidence at hand indicates that disturbances in α_R and α_S become larger as the sample size increases under the particular type of misspecification considered here.

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TABLE 1. Run-length distributions of R charts using $\alpha = 0.05$, samples of size $n = 10$, and the sample variance from a base period of m observations, under normality when the process is in control.

m	$P(N_R > 5)$	$P(N_R > 10)$	$P(N_R > 20)$	$P(N_R > 50)$	$P(N_R > 100)$
5	0.90106	0.85295	0.79521	0.74538	0.73882
10	0.85438	0.79618	0.72168	0.61366	0.55701
20	0.82050	0.72422	0.60897	0.45363	0.33984
40	0.79660	0.66909	0.51149	0.30223	0.17738
∞	0.77378	0.59873	0.35848	0.07694	0.00592

TABLE 2. Run-length distributions of S^2 charts using $\alpha = 0.05$, samples of size $n = 10$, and the sample variance from a base period of m observations, under normality when the process is in control.

m	$P(N_S > 5)$	$P(N_S > 10)$	$P(N_S > 20)$	$P(N_S > 50)$	$P(N_S > 100)$
5	0.90076	0.87008	0.83509	0.80584	0.79672
10	0.86113	0.80833	0.74950	0.67011	0.61025
20	0.82715	0.74039	0.63967	0.50024	0.40076
40	0.80168	0.68204	0.53659	0.34162	0.21937
∞	0.77378	0.59873	0.35848	0.07694	0.00592

TABLE 3. Run-length distributions for the R charts of stationary Gaussian processes having the parameters $\gamma(\xi) = (\xi^2, \xi^2, \ldots)$, using the sample variance in a base period of $m = 10$ observations to estimate σ_0^2, with $\alpha = 0.05$ and samples of size $n = 10$.

ξ	$P(N_R > 5)$	$P(N_R > 10)$	$P(N_R > 20)$	$P(N_R > 50)$	$P(N_R > 100)$
1.0	0.85438	0.79618	0.72168	0.61366	0.55701
1.2	0.67862	0.57321	0.47783	0.35281	0.26283
1.5	0.38038	0.25958	0.16908	0.09150	0.05167
2.0	0.08241	0.03287	0.01212	0.00282	0.00099

TABLE 4. Run-length distributions for the S^2 charts of stationary Gaussian processes having the parameters $\gamma(\xi) = (\xi^2, \xi^2, \ldots)$, using the sample variance in a base period of $m = 10$ observations to estimate σ_0^2, with $\alpha = 0.05$ and samples of size $n = 10$.

ξ	$P(N_S > 5)$	$P(N_S > 10)$	$P(N_S > 20)$	$P(N_S > 50)$	$P(N_S > 100)$
1.0	0.86113	0.80833	0.74950	0.67011	0.61025
1.2	0.68704	0.59400	0.50344	0.39644	0.32490
1.5	0.37996	0.26980	0.18574	0.10983	0.07304
2.0	0.07654	0.03249	0.01315	0.00386	0.00151

TABLE 5. Values of $P(R/\sigma_0 \leq c_\alpha)$ for spherical Gaussian and Cauchy processes using samples of size n.

Gaussian Process	Cuchy Process n = 3	n = 5	n = 10
0.900	0.5786	0.5150	0.4633
0.950	0.6234	0.5548	0.4978
0.975	0.6570	0.5858	0.5270
0.990	0.6899	0.6177	0.5585

TABLE 6. Values of $P(S^2/\sigma_0^2 \leq d_\alpha)$ for spherical Gaussian and Cauchy processes using samples of size n.

Gaussian Process	Cauchy process n = 3	n = 5	n = 10
0.900	0.5778	0.5130	0.4538
0.950	0.6218	0.5515	0.4843
0.975	0.6546	0.5814	0.5089
0.990	0.6870	0.6123	0.5353

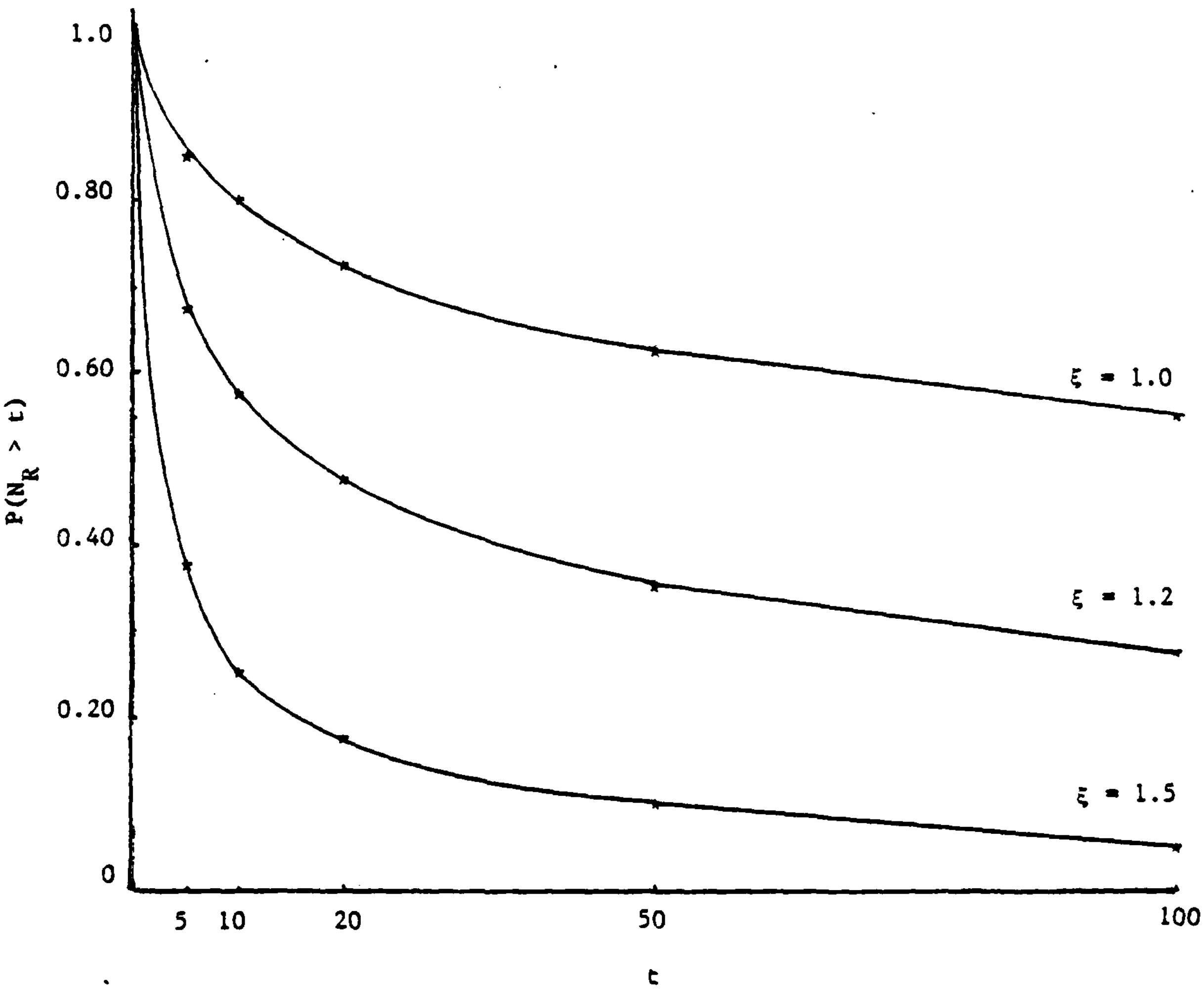

Figure 1. Envelopes for the run lengths of R charts of drifting Gaussian processes in terms of stationary processes having the parameters $\underset{\sim}{\gamma}(\xi) = (\xi^2, \xi^2, \ldots)$.

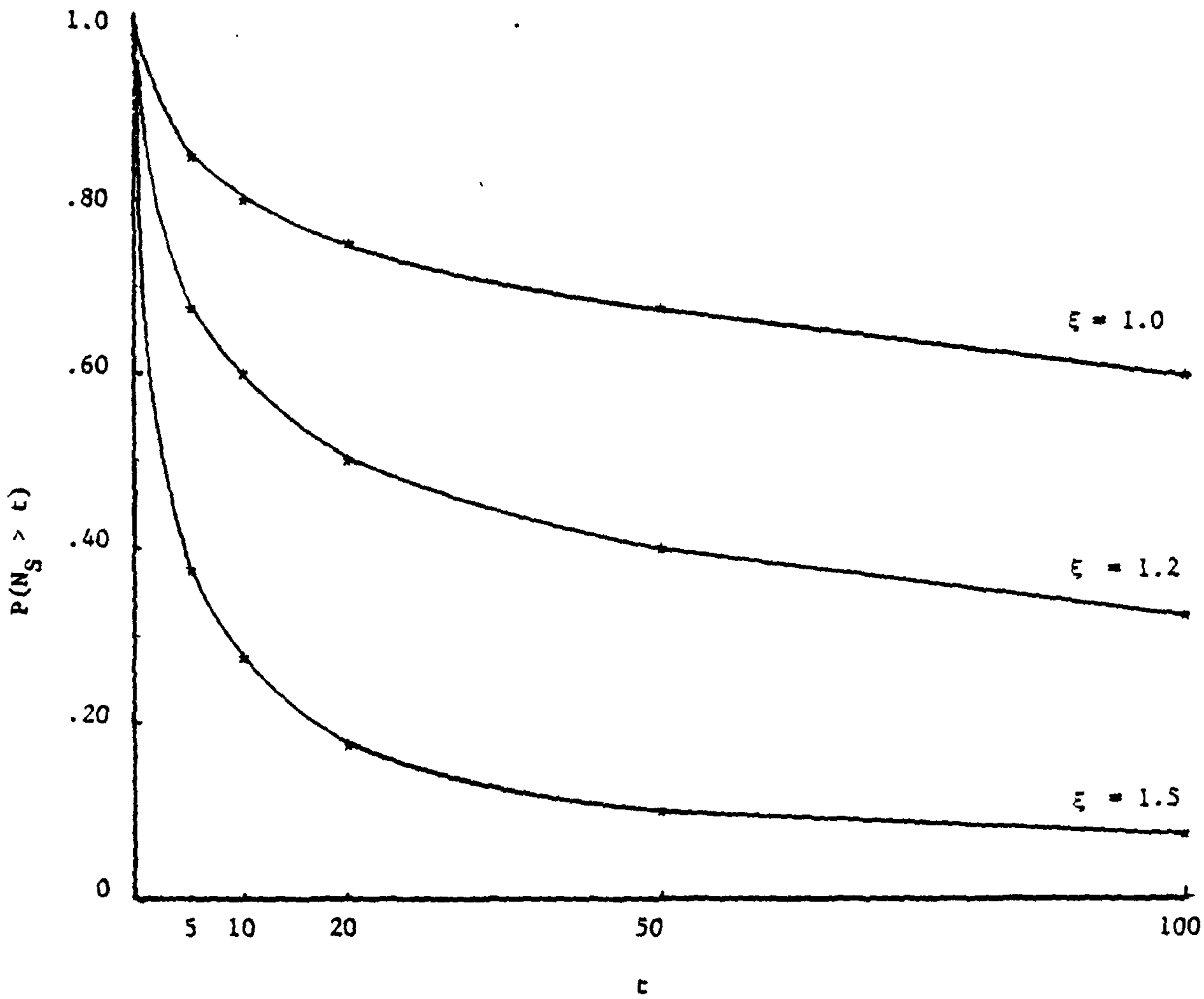

Figure 2. Envelopes for the run lengths of S^2 charts of drifting Gaussian processes in terms of stationary processes having the parameters $\underset{\sim}{\gamma}(\xi) = (\xi^2, \xi^2, \ldots)$.

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Monitoring process variance, run-length distributions, non-Gaussian errors, spherical processes, stochastic ordering.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

R and S^2 charts for monitoring the variability of a process are studied when the control variance, σ_0^2, is unknown, when observations are non-Gaussian, and when the process drifts. Geometric bounds on run-length distributions for the R and S^2 charts are given for all underlying distributions when σ_0^2 is estimated in a base period. Similar bounds apply in the case of a drifting

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process, the notion of which is developed rigorously. It is shown that normal-theory properties of these charts hold exactly for every spherical process when a suitable estimate for σ_0^2 is used. Run-length distributions otherwise are shown to be ordered stochastically given a peakedness ordering on the underlying spherical processes.